

# Kinetic derivation of generalized phase space Chern-Simons theory

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## Abstract

We study a kinetic theory in  $2d$  phase space when all abelian Berry curvatures are nonzero. We derive the complete form of the Poisson brackets, and calculate transports induced by Berry curvatures. Then we construct the low-energy effective theory to reproduce the transports. Such an effective theory is given by the Chern-Simons theory in  $1 + 2d$  dimensions. Some implications of the Chern-Simons theory are also discussed.

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*Introduction.* Topological field theory provides us a universal framework to study electromagnetic properties of insulators, or more generally of the system with no low-energy dynamical degrees of freedom in its bulk state to interact with external fields [1–5]. It has long been known that the Bloch state acquires a nontrivial phase called the Berry phase [6], which accounts novel properties of insulators such as the quantized adiabatic pumping current or the quantized change of electric polarization in  $1 + 1$  dimensions [7–9], and the quantum Hall effect in  $1 + 2$  dimensions [10–15]. In the former case, the Bloch state picks up the Berry phase of a closed orbit around the hybrid torus of crystal momentum and adiabatic pumping parameter. In the latter case, the Bloch state picks up the Berry phase of the two dimensional Brillouin zone. Such quantized transports are described by the effective action for electromagnetic gauge potentials and Berry connections [4, 5]. They construct generalized gauge potentials in the phase space, and the effective action is given by the phase space Chern-Simons theory for the generalized gauge potentials [4].

Such an effect of the Berry phase has been incorporated into kinetic theory [15, 16]. The Bloch wave function is deformed in the presence of inhomogeneity such as inhomogeneous ordering [17–19] or strain gradient [20], and its effect to kinetic theory can also be taken into account via the Berry curvature, where that in the mixed space over real space and momentum is introduced [15, 16]. In general, we can construct a kinetic theory with Berry curvatures in  $(1 + 2d)$ -dimensional hybrid space of phase space and time. The correction to transports due to such hybrid space Berry curvatures has been studied by using kinetic theory [21, 22]. It has shown that such transports induced by the  $(1 + 2d)$ -dimensional Berry curvatures can be described by the generalized phase space Chern-Simons theory [23]. However the connection between the kinetic theory and the generalized phase space Chern-Simons theory has not been understood yet as mentioned in Ref. [23]. As the dimension of system increases, the phase space Berry curvatures more contribute to electromagnetic properties of materials. Thanks to the experimental discovery of topological insulators [24–30], or Dirac/Weyl semimetals [31–35], we can consider the interplay of Berry curvatures in a realistic system. It is important to derive a topological field theory for low energy responses to the Berry curvatures from a firm basis of transport theory, namely, from kinetic theory. It is also meaningful to study higher dimensional system ( $d \geq 4$ ) [36, 37] thanks to the invention of synthetic dimensions [38].

In this Letter, we study a kinetic theory in the case that all abelian Berry curvatures in

$1 + 2d$  dimensions are nonzero. When all  $(1 + 2d)$ -dimensional Berry curvatures are nonzero, the Poisson brackets and the phase space volume have been calculated only in the leading order of the derivative expansion, which is enough to calculate transports in this order. We derive the full form of them, and calculate transports induced by the Berry curvatures. Then, we construct a low energy effective theory to reproduce the transports obtained from the kinetic theory. We show that the effective theory is, in fact, the generalized phase space Chern-Simons theory given in Ref. [23]. We identify the energy  $\varepsilon$  and momentum  $p_i$  as gauge potentials ( $A_t = -\varepsilon$  and  $A_i = p_i$ ) as well as electromagnetic ones and Berry connections, and construct the phase space Chern-Simons theory with all those three gauge potentials. Some implications of it including nonlinear responses are also discussed.

*Kinetic theory in the presence of Berry connections.* We consider the semiclassical dynamics in  $(1 + 2d)$ -dimensional phase space and time. The action has the form [15]:

$$S = \int dt \left( \dot{\mathbf{x}} \cdot (\mathbf{p} + \mathbf{A}) + \dot{\mathbf{p}} \cdot \mathbf{a} - \varepsilon + A_t \right), \quad (1)$$

where  $\varepsilon$  is an energy eigenvalue of a Bloch state  $|u\rangle$ , which can be labeled by time  $t$  and phase space coordinates  $\mathbf{x} = (x_1, \dots, x_d)$  and  $\mathbf{p} = (p_1, \dots, p_d)$ .  $A_t$ ,  $\mathbf{A} = (A_1, \dots, A_d)$ , and  $\mathbf{a} = (a_1, \dots, a_d)$  are the Berry connections and explicitly given as  $A_t = \langle u | i\partial_t | u \rangle$ ,  $\mathbf{A} = \langle u | i\nabla_{\mathbf{x}} | u \rangle$ , and  $\mathbf{a} = \langle u | i\nabla_{\mathbf{p}} | u \rangle$ . At finite external electromagnetic fields, their gauge potentials are added to the action (1), but they can be absorbed into  $A_t$  and  $\mathbf{A}$ . In the following, we do not distinguish electromagnetic gauge potentials and Berry connections unless otherwise stated. We assume that all Berry connections fully depend on  $t$ ,  $\mathbf{x}$ , and  $\mathbf{p}$ . We employ the Einstein convention for repeated indices. We introduce the phase space and time coordinates and the generalized connections as  $\xi_\mu = (\xi_0, \xi_a) = (t, \xi_a)$  and  $\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_a) = (\mathcal{A}_t, \mathcal{A}_a)$  with  $\xi_a = (\mathbf{x}, \mathbf{p})$ ,  $\mathcal{A}_t = -\varepsilon + A_t$  and  $\mathcal{A}_a = (\mathbf{p} + \mathbf{A}, \mathbf{a})$ . We absorb the energy  $\varepsilon$  and momentum  $\mathbf{p}$  into gauge potentials [23], which is different from the standard description of kinetic theory [15, 16]. Using these variables, we can write the action as the topological form:

$$S = \int \mathcal{A}_\mu d\xi_\mu. \quad (2)$$

As will be seen below, this topological nature implies that the low-energy effective field theory for transports is expressed as a  $(1 + 2d)$ -dimensional Chern-Simons theory for  $\mathcal{A}_\mu$ .

The equation of motion reads

$$\omega_{ab}\dot{\xi}_b = -\omega_{at} = \frac{\partial H}{\partial \xi_a} + \partial_t \mathcal{A}_a, \quad (3)$$

where  $H = -\mathcal{A}_t = \varepsilon - A_t$ , and we used  $\dot{\mathcal{A}}_a = \partial_t \mathcal{A}_a + \dot{\xi}_b \partial_{\xi_b} \mathcal{A}_a$ .  $\omega_{ab}$  and  $\omega_{at}$  are the generalized Berry curvatures in phase space and time, and defined as

$$\omega_{ab} = \frac{\partial \mathcal{A}_b}{\partial \xi_a} - \frac{\partial \mathcal{A}_a}{\partial \xi_b}, \quad (4)$$

$$\omega_{at} = \frac{\partial \mathcal{A}_t}{\partial \xi_a} - \frac{\partial \mathcal{A}_a}{\partial t}. \quad (5)$$

We assume that  $\det(\omega)$  is nonzero, and then, the equation of motion (3) is written as

$$\dot{\xi}_a = -\omega_{ab}^{-1} \omega_{bt}. \quad (6)$$

When we fix the gauge to satisfy  $\partial_t \mathcal{A}_a = 0$ , this can be expressed as the Hamilton equation,

$$\dot{\xi}_a = \{\xi_a, H\}_p = \{\xi_a, \xi_b\}_p \frac{\partial H}{\partial \xi_b}, \quad (7)$$

where  $\{\cdot, \cdot\}_p$  represents the Poisson bracket. From Eqs. (6), and (7), the Poisson bracket reads  $\{\xi_a, \xi_b\}_p = \omega_{ab}^{-1}$ . At nonzero Berry curvatures,  $\omega_{ab}^{-1}$ , in general, does not satisfy the standard canonical relations, so that  $x_i$  and  $p_i$  are no longer canonical pairs. Also the invariant phase space volume element is modified as  $d^d x d^d p \sqrt{\det(\omega)}/(2\pi)^d$  [15]. Since  $\omega$  is a skew symmetric matrix, whose determinant is written as  $\det(\omega) = \text{Pf}(\omega)^2$  by using the Pfaffian  $\text{Pf}(\omega) = \epsilon_{a_1 \dots a_{2d}} \omega_{a_1 a_2} \dots \omega_{a_{2d-1} a_{2d}} / 2^d d!$ , where  $\epsilon_{a_1 \dots a_{2d}}$  is the totally anti-symmetric tensor ( $\epsilon_{12 \dots 2d} = 1$ ). We note that the determinant of a real skew matrix is always nonnegative. We can write the Pfaffian as

$$\text{Pf}(\omega) = \frac{1}{2^{d-1}(d-1)!} \epsilon_{aba_1 \dots a_{2d-2}} \omega_{ab} \omega_{a_1 a_2} \dots \omega_{a_{2d-3} a_{2d-2}}, \quad (8)$$

where the summation over  $a$  is not performed. This corresponds to the cofactor expansion of the Pfaffian. We find that the inverse matrix is given as

$$\omega_{ab}^{-1} = \frac{1}{2^{d-1}(d-1)! \text{Pf}(\omega)} \epsilon_{baa_1 \dots a_{2d-2}} \omega_{a_1 a_2} \dots \omega_{a_{2d-3} a_{2d-2}}. \quad (9)$$

Let us check  $\omega_{ca}\omega_{ab}^{-1} = \delta_{cb}$ . If  $c = b$ , from Eqs. (8) and (9),  $\omega_{ca}\omega_{ab}^{-1}$  equals unity. If  $c \neq b$ ,  $\text{Pf}(\omega)\omega_{ca}\omega_{ab}^{-1}$  becomes the Pfaffian of another skew symmetric matrix  $\tilde{\omega}$ , which is obtained from  $\omega$  by replacing the  $b$ -th row of  $\omega$  by its  $c$ -th row. The determinant of  $\tilde{\omega}$  must be zero since two rows ( $b$ - and  $c$ -th rows) are the same by its construction. Since  $\det(\tilde{\omega}) = \text{Pf}(\tilde{\omega})^2 = (\text{Pf}(\omega)\omega_{ca}\omega_{ab}^{-1})^2$ ,  $\omega_{ca}\omega_{ab}^{-1} = 0$  (we assume that  $\det(\omega) \neq 0$ ), and thus,  $\omega_{ca}\omega_{ab}^{-1} = \delta_{cb}$ . Although the Poisson brackets are in general given by Eq. (9), it is useful for practical applications to show their explicit forms, so that we write down them in the case of  $d = 2, 3$  in supplemental material.

By using Eqs. (6) and (9), we can calculate transports. Under weak external fields, the transports can be described by the Boltzmann equation,

$$\partial_t n(t, \xi_a) + \dot{\xi}_a \partial_{\xi_a} n(t, \xi_a) = 0, \quad (10)$$

where  $n(t, \xi_a)$  is the distribution function in  $(1 + 2d)$ -dimensional phase space and time, and we neglect collision terms.  $\xi_a$  is the solutions of the equation of motion (6) and given as

$$\sqrt{\det(\omega)} \dot{\xi}_a = \frac{(-1)^\nu}{2^{d-1}(d-1)!} \epsilon_{aba_1 \dots a_{2d-2}} \omega_{a_1 a_2} \dots \omega_{a_{2d-1} a_{2d-2}} \omega_{bt}, \quad (11)$$

where we introduce the sign of the Pfaffian  $(-1)^\nu = \text{Pf}(\omega)/\sqrt{\det(\omega)}$ . In our kinetic regime, the sign is negative (see the supplemental material for the detail.) The current can be calculated by averaging the velocity of quasi particles,  $\dot{\xi}_a$ , over the phase space with the modified volume  $\sqrt{\det(\omega)}/(2\pi)^d$ . The current density is given as

$$\begin{aligned} j_a &= \frac{1}{(2\pi)^d} \sqrt{\det(\omega)} \dot{\xi}_a n(t, \xi) \\ &= \frac{(-1)^\nu}{(2\pi)^d 2^{d-1} (d-1)!} \epsilon_{aba_1 \dots a_{2d-2}} \omega_{a_1 a_2} \dots \omega_{a_{2d-1} a_{2d-2}} \omega_{bt} n(t, \xi). \end{aligned} \quad (12)$$

Integrating Eq. (12) with respect to  $x_i$  and/or  $p_i$ , we obtain the current in real, momentum or mixed spaces. Since the Boltzmann equation (10) is symmetric in  $x_i$  and  $p_i$ , we can also consider the conservation law in momentum space, in which the force term  $\sqrt{\omega} \dot{p}_i n(t, \mathbf{p})$  plays a role of the current. Equation (12) can be applied to not only near equilibrium but also far from equilibrium as long as the kinetic theory is applicable. For example, we can calculate Berry curvature corrections to dissipative currents in nonequilibrium steady state

with using the relaxation time approximation.

The local number density is given as  $j_0 = \sqrt{\det(\omega)}n(t, \xi)/(2\pi)^d$ . If we consider a band insulator ( $n(t, \xi_a) = 1$ ), we have

$$j_0(\xi) = \sqrt{\det(\omega)}/(2\pi)^d. \quad (13)$$

This is the local induced density, when the Berry curvatures are adiabatically introduced. For the explicit form of Eqs. (12) and (13) in  $d = 2, 3$ , see supplemental material.

*Topological effective field theory.* We here construct a low-energy effective theory to describe the reactions to Berry curvatures in Eqs. (12) and (13) ( $n(t, \xi_a) = 1$ ). We find that such an effective theory is represented by the phase space Chern-Simons theory [23], whose action is

$$S_{\text{CS}} = \frac{(-1)^\nu}{(2\pi)^d(d+1)!} \int dt d^{2d} \xi \epsilon_{\mu_0 \dots \mu_{2d}} \mathcal{A}_{\mu_0} \partial_{\mu_1} \mathcal{A}_{\mu_2} \dots \partial_{\mu_{2d-1}} \mathcal{A}_{\mu_{2d}}, \quad (14)$$

where  $\mu_i = 0, 1, \dots, 2d$ ,  $\partial_{\mu_i} = (\partial_t, \partial_{\xi_1} \dots, \partial_{\xi_{2d}})$ ,  $\mathcal{A}_\mu = (\mathcal{A}_t, \mathcal{A}_a) = (-\varepsilon + A_t, p_i + A_i, a_i)$ . The current density is obtained by differentiating the action (14) with respect to  $\mathcal{A}_\mu$  as

$$j_\mu = \frac{\partial S_{\text{CS}}}{\partial \mathcal{A}_\mu} = \frac{(-1)^\nu}{(2\pi)^d 2^d d!} \epsilon_{\mu \mu_1 \dots \mu_{2d}} \omega_{\mu_1 \mu_2} \dots \omega_{\mu_{2d-1} \mu_{2d}}, \quad (15)$$

where  $\omega_{\mu_i \mu_j} = \partial_{\mu_i} \mathcal{A}_{\mu_j} - \partial_{\mu_j} \mathcal{A}_{\mu_i}$ . Equation (15) recovers Eqs. (12) and (13) ( $n(t, \xi_a) = 1$ ). This is our main result. We see that all currents obtained from kinetic theory can be expressed by the Chern-Simons current (15), so that two descriptions are completely the same. Below we discuss some interesting results obtained from Eq. (15). The following argument can be applied to both of fermion and boson. This is because if we can prepare a uniformly filled band for bosons, the Boltzmann equation (10) does not distinguish particle statistics without collision terms. In fact such a band occupation of bosons is experimentally realized in ultracold atom systems [39].

First we consider the adiabatic pumping current or equivalently the change of polarization in the presence of time-periodic perturbation. We consider a single band insulator for simplicity. The distribution reads  $n(t, \xi_a) = 1$ . The change of polarization is obtained by integrating the current over the period of perturbation  $T$  as, in  $d = 2$ ,

$$P_i(x_i) = -e \int_0^T dt j_i(x_i) = e \int \frac{dt d^2 p}{(2\pi)^2} [\delta_{ij} \Omega_{p_j t} + (\delta_{ij} \Omega_{p_k x_k} - \Omega_{p_i x_j}) \Omega_{p_j t} + \Omega_{p_i p_j} \Omega_{x_j t}], \quad (16)$$

where  $e > 0$  is the electric charge,  $\Omega_{p_j t} = \partial_{p_j} A_t - \partial_t a_j$ ,  $\Omega_{p_i x_j} = \partial_{p_i} A_j - \partial_{x_i} a_j$ ,  $\Omega_{p_i p_j} = \partial_{p_i} a_j - \partial_{p_j} a_i$ , and  $\Omega_{x_j t} = \partial_{x_j} A_t - \partial_t A_j$ . The first term is the adiabatic charge pumping given by Thouless [7]. The second and third terms are corrections in the presence of spatial inhomogeneity [22]. Equation (16) completely reproduces the result of Ref. [22]. There exist no higher order correction in  $d = 2$ . In contrast, in  $d = 3$ , we find

$$P_i(x_i) = e \int \frac{dt d^3 p}{(2\pi)^3} \left[ (\delta_{ij} + \delta_{ij} \Omega_{x_k p_k} - \Omega_{p_i x_j} - \Omega_{x_i} \Omega_{p_j} + \epsilon_{ikl} \epsilon_{j\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2) \Omega_{p_j t} \right. \\ \left. + (\epsilon_{ijk} \Omega_{p_k} + \epsilon_{ijk} \Omega_{p_{\bar{k}} x_k} \Omega_{p_{\bar{k}}}) \Omega_{x_j t} \right], \quad (17)$$

where  $\Omega_{x_i} = \epsilon_{ijk} \Omega_{x_j x_k} / 2$ , and  $\Omega_{p_i} = \epsilon_{ijk} \Omega_{p_j p_k} / 2$ . This expression recovers the result of Ref. [22] in the first order of the spatial gradient, and the higher order terms are derived in Ref. [23].

Since the Chern-Simons action is symmetric in  $x_i$  and  $p_i$ , we can consider a momentum analogue of topological currents in real space. As an example, let us consider the quantum Hall effect. In  $d = 2$ , the Chern-Simons action (14) includes

$$S_{\text{CS}} = \frac{(-1)^\nu}{8\pi^2} \int dt d^2 x d^2 p \epsilon_{\mu_1 \mu_2 \mu_3} A_{\mu_1} \partial_{\mu_2} A_{\mu_3} \Omega_{p_1 p_2}, \quad (18)$$

where  $\mu_i = 0, 1, 2$ ,  $\partial_{\mu_i} = (\partial_t, \partial_{x_1}, \partial_{x_2})$ , and  $A_\mu = (A_t, A_x, A_y)$ , and  $\Omega_{p_1 p_2} = \partial_{p_1} a_{p_2} - \partial_{p_2} a_{p_1}$  is the conventional Berry curvature of a Bloch band. If  $A_i$  are gauge potentials for external electromagnetic fields, and  $\Omega_{p_1 p_2}$  does not depend on  $x_i$ , Eq. (18) reads

$$S_{\text{CS}} = \frac{C_1}{4\pi} \int dt d^2 x \epsilon_{\mu_1 \mu_2 \mu_3} A_{\mu_1} \partial_{\mu_2} A_{\mu_3}, \quad (19)$$

where  $C_1 = (-1)^\nu \int d^2 p \Omega_{p_1 p_2} / 2\pi$  is the first Chern number and takes only integer values. This is the Chern-Simons theory for the integer quantum Hall effect. In addition, the Chern-Simons action (14) includes

$$S_{\text{CS}} = \frac{(-1)^\nu}{8\pi^2} \int dt d^2 x d^2 p \Omega_{x_1 x_2} \epsilon_{\mu_1 \mu_2 \mu_3} a_{\mu_1} \partial_{\mu_2} a_{\mu_3}, \quad (20)$$

where  $\mu_i = 0, 1, 2$ ,  $\partial_{\mu_i} = (\partial_t, \partial_{p_1}, \partial_{p_2})$ , and  $a_\mu = (A_t, a_{p_1}, a_{p_2})$ , and  $\Omega_{x_1 x_2} = \partial_{x_1} A_{x_2} - \partial_{x_2} A_{x_1}$ .

If  $\Omega_{x_1x_2}$  does not depend on  $p_i$ , Eq. (20) reads

$$S_{\text{CS}} = \frac{C_p(-1)^\nu}{4\pi} \int dt d^2p \epsilon_{\mu_1\mu_2\mu_3} a_{\mu_1} \partial_{\mu_2} a_{\mu_3}, \quad (21)$$

with  $C_p = \int d^2x \Omega_{x_1x_2}/2\pi$ . This is the magnetic flux perpendicular to the two dimensional system. In the presence of topological defects with a magnetic charge such as a skyrmion,  $C_p$  is quantized; the Chern-Simons action (21) describes the integer quantum Hall effect in momentum space. We can physically understand this effect through the adiabatic pumping. Let us consider a time-periodic adiabatic perturbation. Since the current in momentum space is the acceleration, the time and momentum integration of the quantum Hall current,  $j_{p_i} = (C_p/2\pi)\epsilon_{ij}\Omega_{p_jt}$ , over a period of cycle and Brillouin zone gives a contribution to the change of total momentum over one cycle. In fact, there is the leading contribution, which is a momentum analogue of the Thoules pumping [40]. Equation (21) gives the correction to the adiabatic momentum pumping like the last term in Eq. (16). The full form of the momentum pumping is given by Eqs. (16) and (17) with the exchange of  $x$  and  $p$ .

*Concluding remarks.* We have calculated transports when all  $(1+2d)$ -dimensional abelian Berry curvatures are nonzero on the basis of the kinetic theory and the phase space Chern-Simons theory. To calculate the transports by using the kinetic theory, we have derived the exact form of Poisson brackets. By using the Poisson brackets, we have analyzed transports induced via Berry curvatures. Then, we have shown that all transports calculated from the kinetic theory can be described by the generalized phase space Chern-Simons theory. We could have shown a clear connection between the kinetic theory and the phase space Chern-Simons theory, which was not understood in a pervious work [23]. Our work also provides the clear derivation of the generalized phase space Chern-Simons theory, not based on the analogy between the phase space theory and the noncommutative lowest Landau projected theory in  $2d$  real space [23]. The explicit form of transports, which are important for practical purpose, are summarized in supplemental materials.

We comment on the fact that the Chern-Simons current (15) includes nonlinear responses to Berry curvatures, which are higher  $\hbar$  order terms, and in general, suffer from the next leading order corrections in the view of kinetic theory [22]. Such corrections will disappear provided that the kinetic theory still keeps the topological form like Eq. (2) when the high order terms in the derivative expansion are taken into account, namely, provided that the



low-energy theory has area-preserving diffeomorphism invariance in phase space. In the presence of it, corrections are restricted to the form, which can be described by the phase space Chern-Simons theory. However, we do not know the general condition for the system to have phase-space area-preserving diffeomorphism invariance as a low-energy emergent symmetry. To clarify the condition is an important future study.

There are several generalizations of our work. One is that we can consider degenerate Bloch states and non-abelian Berry curvatures. In this case, a non-abelian version of the generalized phase-space Chern-Simons theory would give us the inclusive description of electromagnetic properties, which should include the unified theory of topological insulators given in Ref. [4]. Another direction is a generalization to include other transports such as thermal or spin currents [41–44]. It is also interesting to consider the dynamical gauge fields in phase space. We assume that all gauge potentials in phase space are static in this Letter, but it could be dynamical if it arises from the corrective behavior of interacting many-body electrons. In fact, such a dynamical gauge fields would emerge when we consider an low-energy effective theory of the fractional quantum Hall state [1–3]. It will be interesting to see the behavior of the effective theory obtained by integrating out the dynamical gauge fields from the generalized phase space Chern-Simons theory as conducted in the fractional quantum Hall effect.

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**Supplemental Material for**  
**“Generalized phase space Chern-Simons theory in  $1 + 2d$  dimensions”**

**Poisson brackets in  $1 + 2$  dimensions**

Here we derive the Poisson brackets in  $d = 2$  by assuming that all abelian Berry curvatures become nonzero. We use the standard notation to compare the results with previous works. For the phase space coordinates  $\xi_a = (x_i, p_i) = (x_1, x_2, p_1, p_2)$ , the equation of motion reads

$$(\Omega - J)_{ab} \dot{\xi}_b = \frac{\partial H}{\partial \xi_a} + \partial_t \mathcal{A}_a, \quad (22)$$

where  $H = \varepsilon - A_t$ , and  $J_{ab}$  is an anti-symmetric matrix:

$$J_{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (23)$$

with  $I$  being the  $2 \times 2$  unit matrix.  $\Omega_{ab}$  is the Berry curvature and defined as

$$\Omega_{ab} = \frac{\partial \mathcal{A}_b}{\partial \xi_a} - \frac{\partial \mathcal{A}_a}{\partial \xi_b}, \quad (24)$$

with  $\mathcal{A}_a = (\mathbf{A}_i, \mathbf{a}_i)$ . We assume  $\det(\Omega - J) \neq 0$ . Then the equation of motion reads

$$\dot{\xi}_a = (\Omega - J)_{ab}^{-1} \left( \frac{\partial H}{\partial \xi_b} + \partial_t \mathcal{A}_b \right). \quad (25)$$

From this equation, the Poisson brackets are

$$\{\xi_a, \xi_b\}_p = (\Omega - J)_{ab}^{-1}. \quad (26)$$

The invariant phase space volume element is modified as  $d^2 x d^2 p \sqrt{\det(\Omega - J)}/(2\pi)^2$ . In  $1 + 2$  dimensions,  $\Omega - J$  is given explicitly as

$$(\Omega - J)_{ab} = \begin{pmatrix} \Omega_{x_i x_j} & -\delta_{ij} - \Omega_{p_j x_i} \\ \delta_{ij} + \Omega_{p_i x_j} & \Omega_{p_i p_j} \end{pmatrix}, \quad (27)$$

where  $i, j = 1, 2$ . Its inverse matrix is

$$(\Omega - J)_{ab}^{-1} = \frac{(-1)^\nu}{\sqrt{\det(\Omega - J)}} \begin{pmatrix} -\Omega_{p_i p_j} & -\delta_{ij}(1 + \Omega_{p_k x_k}) + \Omega_{p_i x_j} \\ \delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_j x_i} & -\Omega_{x_i x_j} \end{pmatrix}. \quad (28)$$

In the kinetic regime  $\Omega \ll 1$ , the sign of the Pfaffian is negative:  $(-1)^\nu = \text{Pf}(\Omega - J)/\sqrt{\det(\Omega - J)} \simeq \text{Pf}(-J) = -1$ , and the Jacobian of  $\Omega - J$  reads

$$\sqrt{\det(\Omega - J)} = 1 + \Omega_{p_i x_i} - \epsilon_{abcd} \Omega_{\xi_a \xi_b} \Omega_{\xi_c \xi_d} / 8, \quad (29)$$

where  $\epsilon_{abcd}$  is the totally anti-symmetric tensor with  $\epsilon_{1234} = 1$  ( $\xi_a = (\xi_1, \xi_2, \xi_3, \xi_4)$ ).

From Eq. (28), the Poisson brackets are expressed as

$$\{x_i, x_j\}_p = \Omega_{p_i p_j} / \sqrt{\det(\Omega - J)}, \quad (30)$$

$$\{x_i, p_j\}_p = (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j}) / \sqrt{\det(\Omega - J)}, \quad (31)$$

$$\{p_i, x_j\}_p = -(\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_j x_i}) / \sqrt{\det(\Omega - J)}, \quad (32)$$

$$\{p_i, p_j\}_p = \Omega_{x_i x_j} / \sqrt{\det(\Omega - J)}. \quad (33)$$

From Eqs. (25) and (28), we have

$$\sqrt{\det(\Omega - J)} \dot{x}_i = (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j}) \tilde{v}_j - \Omega_{p_i p_j} \tilde{E}_j, \quad (34)$$

$$\sqrt{\det(\Omega - J)} \dot{p}_i = (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_j x_i}) \tilde{E}_j + \Omega_{x_i x_j} \tilde{v}_j, \quad (35)$$

where  $\tilde{v}_i = \partial_{p_i} \varepsilon - \Omega_{p_i t}$ , and  $\tilde{E}_i = \Omega_{x_i t} - \partial_{x_i} \varepsilon$  [15]. At finite magnetic fields perpendicular to the two-dimensional system ( $\Omega_{x_1 x_2} \neq 0$ ), the energy of a quasi-particle has a correction due to its nonzero magnetic moment, which is given as  $\varepsilon = \varepsilon_0 \left(1 - \frac{q}{\hbar c} \Omega_{x_1 x_2} \Omega_{p_1 p_2}\right)$  [15] with  $\varepsilon_0$  being the energy at zero magnetic field. When the magnetic field is spatially non-uniform, it induces an effective electric field through  $\partial_{x_i} \varepsilon$ . For a band insulator,  $n(t, \xi_a) = 1$ , we can calculate the transports in real space by integrating Eq. (34) over momentum space as

$$j_i = \int \frac{d^2 p}{(2\pi)^2} \left[ (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j}) \tilde{v}_j - \Omega_{p_i p_j} \tilde{E}_j \right]. \quad (36)$$

## Transports in 1 + 3 dimensions

Here we derive the Poisson brackets in  $1 + 3$  dimensions by assuming that all abelian Berry curvatures become nonzero. In  $1 + 3$  dimensions,  $\Omega - J$  reads

$$(\Omega - J)_{ab} = \begin{pmatrix} \Omega_{x_i x_j} & -\delta_{ij} - \Omega_{p_j x_i} \\ \delta_{ij} + \Omega_{p_i x_j} & \Omega_{p_i p_j} \end{pmatrix}, \quad (37)$$

where  $i, j = 1, 2, 3$ . Its inverse matrix is given as (see also the Poisson brackets shown below)

$$\sqrt{\det(\Omega - J)} (\Omega - J)_{ab}^{-1} = \begin{pmatrix} \epsilon_{ijk} \Omega_{p_k} + \epsilon_{ijk} \Omega_{p_{\bar{k}} x_k} \Omega_{p_{\bar{k}}} & +\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j} - \Omega_{x_i} \Omega_{p_j} + \epsilon_{ikl} \epsilon_{j\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2 \\ -\delta_{ij}(1 + \Omega_{p_k x_k}) + \Omega_{p_j x_i} + \Omega_{x_j} \Omega_{p_i} - \epsilon_{jkl} \epsilon_{i\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2 & \epsilon_{ijk} \Omega_{x_k} + \epsilon_{ij\bar{k}} \Omega_{p_{\bar{k}} x_l} \Omega_{x_l} \end{pmatrix}, \quad (38)$$

where  $\Omega_{x_i} = \epsilon_{ijk} \Omega_{x_j x_k} / 2$ , and  $\Omega_{p_i} = \epsilon_{ijk} \Omega_{p_j p_k} / 2$ .  $\epsilon_{ijk}$  is the totally anti-symmetric tensor with  $\epsilon_{123} = 1$ . We used the fact that  $(-1)^\nu = -1$ . The Jacobian of  $\Omega - J$  reads

$$\begin{aligned} \sqrt{\det(\Omega - J)} &= 1 + \Omega_{p_i x_i} - \Omega_{x_i} \Omega_{p_i} + ((\Omega_{p_i x_i})^2 - \Omega_{p_i x_j} \Omega_{p_j x_i}) / 2 \\ &\quad - \Omega_{p_i} \Omega_{p_i x_j} \Omega_{x_j} + \epsilon_{ijk} \epsilon_{l\bar{m}\bar{n}} \Omega_{p_{\bar{l}} x_i} \Omega_{p_{\bar{m}} x_j} \Omega_{p_{\bar{n}} x_k} / 6. \end{aligned} \quad (39)$$

The detail of the Jacobian is not important to calculate transports since we only need numerators of the Poisson brackets shown below. The Poisson brackets are

$$\{x_i, x_j\}_p = (\epsilon_{ijk} \Omega_{p_k} + \epsilon_{ijk} \Omega_{p_{\bar{k}} x_k} \Omega_{p_{\bar{k}}}) / \sqrt{\det(\Omega - J)}, \quad (40)$$

$$\{x_i, p_j\}_p = (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j} - \Omega_{x_i} \Omega_{p_j} + \epsilon_{ikl} \epsilon_{j\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2) / \sqrt{\det(\Omega - J)}, \quad (41)$$

$$\{p_i, x_j\}_p = -(\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_j x_i} - \Omega_{x_j} \Omega_{p_i} + \epsilon_{jkl} \epsilon_{i\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2) / \sqrt{\det(\Omega - J)}, \quad (42)$$

$$\{p_i, p_j\}_p = (\epsilon_{ijk} \Omega_{x_k} + \epsilon_{ij\bar{k}} \Omega_{p_{\bar{k}} x_l} \Omega_{x_l}) / \sqrt{\det(\Omega - J)}. \quad (43)$$

When  $\Omega_{x_i p_j}$  are set to zero, these results recover the expressions given in Ref. [15]. From

the Poisson brackets (40), (41), (42), and (43), we have

$$\begin{aligned} \sqrt{\det(\Omega - J)}\dot{x}_i &= (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_i x_j} - \Omega_{x_i} \Omega_{p_j} + \epsilon_{ikl} \epsilon_{j\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2) \tilde{v}_j \\ &\quad - (\epsilon_{ijk} \Omega_{p_k} + \epsilon_{ijk} \Omega_{p_{\bar{k}} x_k} \Omega_{p_{\bar{k}}}) \tilde{E}_j \end{aligned} \quad (44)$$

$$\begin{aligned} \sqrt{\det(\Omega - J)}\dot{p}_i &= (\delta_{ij}(1 + \Omega_{p_k x_k}) - \Omega_{p_j x_i} - \Omega_{x_j} \Omega_{p_i} + \epsilon_{jkl} \epsilon_{i\bar{m}\bar{n}} \Omega_{p_{\bar{m}} x_k} \Omega_{p_{\bar{n}} x_l} / 2) \tilde{E}_j \\ &\quad + (\epsilon_{ijk} \Omega_{x_k} + \epsilon_{ijk} \Omega_{p_{\bar{k}} x_l} \Omega_{x_l}) \tilde{v}_j \end{aligned} \quad (45)$$

where  $\tilde{v}_i = \partial_{p_i} \varepsilon - \Omega_{p_i t}$ , and  $\tilde{E}_i = \Omega_{x_i t} - \partial_{x_i} \varepsilon$  [15]. We can calculate transports in real space from  $j_i(t, \mathbf{x}) = \int d^3 p / (2\pi)^3 \sqrt{\det(\Omega - J)} \dot{x}_i n(t, \mathbf{x}, \mathbf{p})$ . Also the change of polarization  $P_i$  is calculated from  $P_i = \int dt \int d^3 x j_i$ , which is discussed in the main text.